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Maximum principle for the generalized time-fractional diffusion equation

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ABSTRACT

In the paper, a maximum principle for the generalized time-fractional diffusion equation over an open bounded domain $G \times (0, T)$, $G \subset R^n$ is formulated and proved. The proof of the maximum principle is based on an extremum principle for the Caputo–Dzherbashyan fractional derivative that is given in the paper, too. The maximum principle is then applied to show that the initial-boundary-value problem for the generalized time-fractional diffusion equation possesses at most one classical solution and this solution continuously depends on the initial and boundary conditions.

1. Introduction

The theory of the derivatives and integrals of a non-integer (fractional) order called *Fractional Calculus* becomes increasingly more important for applications. Both the ordinary and the partial differential equations of fractional order have been used within the last few decades for modeling of many physical and chemical processes and in engineering (see e.g. [2,3,5,7–9,13,14,16,17] and references therein). As stated in [9], partial fractional differential equations became especially important for the modeling of the so called anomalous phenomena in nature and in the theory of the complex systems. The main reason for the utility of the partial differential equations of fractional order like the ones considered in this paper is the strong relationship between these equations and fractional Brownian motion, the continuous time random walk (CTRW) method, the Lévi stable distributions, and the generalized central limit theorem (see e.g. [2,3,7,16]). Moreover, the differential equations of fractional order allow for the representation of the long-memory and non-local dependence of many anomalous processes.

As to the mathematical theory of the differential equations of fractional order, the current situation for the ordinary differential equations is different from the one for the partial differential equations. Whereas it is more or less complete for the ordinary differential equations of fractional order (see e.g. [9,11,12,17]), there are still many gaps in the theory of the partial differential equations of fractional order. In the literature, mainly the initial-value problems for these equations were considered until now. For the methods of solution of such problems we refer the reader to e.g. [4,6,9,10,15,17,19]. As to the boundary-value or initial-boundary-value problems, they were mainly investigated for the case of one spatial variable and/or for the case of the constant coefficients (see e.g. [1,6,18,20]).

In the paper, the uniqueness problem of the solution of the initial-boundary-value problems for the generalized time-fractional diffusion equation over an open bounded domain $G \times (0, T)$, $G \subset R^n$ is considered. This equation is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of the order α $(0 < \alpha \leqslant 1)$ and the second-order spatial derivative by a more general second-order differential operator:

$$\big(D_t^\alpha u\big)(t) = L(u) + F(x,t), \quad 0 < \alpha \leqslant 1, \ (x,t) \in \Omega_T := G \times (0,T), \ G \subset R^n, \tag{1}$$

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where

$$L(u) := \operatorname{div}\big(p(x)\operatorname{grad} u\big) - q(x)u, \quad p \in C^1(\bar{G}),\ q \in C(\bar{G}),\ p(x) > 0,\ q(x) \geqslant 0,\ x \in \bar{G}, \tag{2}$$

the fractional derivative is defined in the Caputo–Dzherbashyan sense

$$\big(D^\alpha f\big)(t) := \big(I^{1-\alpha} f'\big)(t), \quad 0 < \alpha \leqslant 1, \tag{3}$$

I^α being the fractional Riemann–Liouville integral

$$\big(I^\alpha f\big)(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & 0 < \alpha < 1, \\ f(t), & \alpha = 0, \end{cases}$$

and the domain G with the boundary S is open and bounded in R^n.

The operator L is in fact a linear elliptic differential operator of the second order

$$L(u) = \sum_{k=1}^{n} \left(p(x) \frac{\partial^2 u}{\partial x_k^2} + \frac{\partial p}{\partial x_k} \frac{\partial u}{\partial x_k} \right) - q(x)u,$$

that can be represented in the form

$$L(u) = p(x)\Delta u + (\operatorname{grad} p, \operatorname{grad} u) - q(x)u, \tag{4}$$

Δ being the Laplace operator.

If $\alpha = 1$, Eq. (1) coincides with the linear second-order parabolic partial differential equation. The theory of this equation is very well-known, so that we focus in the further discussions on the case $0 < \alpha < 1$.

The rest of the paper is organized as follows. In the second section, the maximum principle for the generalized time-fractional diffusion equation is proved. To prove the maximum principle, an appropriate extremum principle for the Caputo–Dzherbashyan fractional derivative (3) is first established. In the third section, the maximum principle is applied to show that the initial-boundary-value problem under consideration possesses at most one classical solution. This solution—if it exists—depends continuously on the data given in the problem.

2. Maximum principle

Eq. (1) possesses in general an infinite number of solutions. In the real world situations that are modeled with Eq. (1), certain conditions that describe the initial state of the corresponding process and the observations of its visible parts ensure the deterministic character of the process. In the paper, the initial-boundary-value problem

$$u|_{t=0} = u_0(x), \quad x \in \bar{G}, \tag{5}$$

$$u|_S = v(x,t), \quad (x,t) \in S \times [0,T] \tag{6}$$

for Eq. (1) is considered. Here S denotes as usual the boundary of the domain G and $\bar{G}$ its closure.

The first point that has to be clarified in connection with the problem (1), (5), (6) is related to the notion of its solution.

Definition 1. A classical solution of the problem (1), (5), (6) is called a function $u = u(x,t)$ defined in the domain $\bar{\Omega}_T := \bar{G} \times [0,T]$ that belongs to the space $CW_T(G) := C(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ and satisfies both Eq. (1) and the initial and boundary conditions (5)–(6). By $W_t^1((0,T))$ the space of functions $f \in C^1((0,T])$ such that $f' \in L((0,T))$ is denoted, $L((0,T))$ being the set of functions Lebesgue integrable in $(0,T)$.

If the problem (1), (5), (6) possesses a classical solution, then the functions F, u_0 and v given in the problem have to belong to the spaces $C(\Omega_T)$, $C(\bar{G})$ and $C(S \times [0,T])$, respectively. In the further discussions, we always suppose these inclusions to be valid.

In the paper, we deal with the uniqueness of the solution of the problem (1), (5), (6) that is proved by means of an appropriate maximum principle for Eq. (1). In its turn, the proof of the maximum principle is based on an extremum principle for the Caputo–Dzherbashyan fractional derivative (3) that is given in the following theorem.

Theorem 1. *Let a function $f \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum over the interval $[0,T]$ at the point $\tau = t_0$, $t_0 \in (0,T]$. Then the Caputo–Dzherbashyan fractional derivative of the function f is non-negative at the point t_0 for any α, $0 < \alpha < 1$:*

$$\big(D^\alpha f\big)(t_0) \geqslant 0, \quad 0 < \alpha < 1. \tag{7}$$

Let us first introduce an auxiliary function

$$g(\tau) := f(t_0) - f(\tau), \quad \tau \in [0,T],$$

that possesses the following properties:

$$g(\tau) \geqslant 0, \tau \in [0, T], \tag{8}$$

because the function f attains its maximum over the interval $[0, T]$ at the point $\tau = t_0$, $t_0 \in (0, T]$,

$$(D^\alpha g)(t) = -(D^\alpha f)(t), \quad t \in [0, T], \tag{9}$$

because the Caputo–Dzherbashyan fractional derivative is a linear operator and $(D^\alpha C)(t) \equiv 0$, C being a constant,

$$|g(\tau)| \leqslant C_\epsilon |t_0 - \tau|, \quad \tau \in [\epsilon, T], \ 0 < \epsilon < T, \tag{10}$$

because the function f belongs to the space $C^1((0, T])$.

The representation

$$(D^\alpha g)(t_0) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} (t_0 - \tau)^{-\alpha} g'(\tau)\, d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^{\epsilon} (t_0 - \tau)^{-\alpha} g'(\tau)\, d\tau + \frac{1}{\Gamma(1-\alpha)} \int_\epsilon^{t_0} (t_0 - \tau)^{-\alpha} g'(\tau)\, d\tau$$
$$= I_1 + I_2$$

is valid for any fixed value of ϵ, $0 < \epsilon < t_0$. Since $f \in W_t^1((0, T))$, the function g belongs to the space $W_t^1((0, T))$, too, and it means in particular that $g' \in L((0, T))$. The statement

$$\forall \delta > 0 \ \exists \epsilon > 0 \text{ such that } \quad |I_1| \leqslant \delta \tag{11}$$

follows from this last inclusion. As to the second integral, I_2, we use the integration by parts formula and the property (10) of the function g to get the representation

$$I_2 = -\frac{(t_0 - \epsilon)^{-\alpha} g(\epsilon)}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(-\alpha)} \int_\epsilon^{t_0} (t_0 - \tau)^{-\alpha - 1} g(\tau)\, d\tau.$$

The inequality (8) and the fact that $\Gamma(-\alpha) < 0$ for $0 < \alpha < 1$ lead to the relation

$$I_2 \leqslant 0,$$

that together with (9) and (11) finishes the proof of the theorem.

The maximum principle for the generalized time-fractional diffusion equation (1) is given by the following theorem.

Theorem 2. *Let a function $u \in CW_T(G)$ be a solution of the generalized time-fractional diffusion equation (1) in the domain Ω_T and $F(x, t) \leqslant 0$, $(x, t) \in \Omega_T$. Then either $u(x, t) \leqslant 0$, $(x, t) \in \bar{\Omega}_T$ or the function u attains its positive maximum on the bottom or back-side parts $S_G^T := (\bar{G} \times \{0\}) \cup (S \times [0, T])$ of the boundary of the domain Ω_T, i.e.,*

$$u(x, t) \leqslant \max\Big\{0, \max_{(x,t) \in S_G^T} u(x, t)\Big\}, \quad \forall (x, t) \in \bar{\Omega}_T. \tag{12}$$

We first suppose that the statement of the theorem does not hold true, i.e., $\exists (x_0, t_0)$, $x_0 \in G$, $0 < t_0 \leqslant T$ with the property

$$u(x_0, t_0) > \max_{(x,t) \in S_G^T} \{0, u(x, t)\} = M > 0. \tag{13}$$

Let us define the number $\epsilon := u(x_0, t_0) - M > 0$ and introduce the auxiliary function

$$w(x, t) := u(x, t) + \frac{\epsilon}{2} \frac{T - t}{T}, \quad (x, t) \in \bar{\Omega}_T.$$

It follows from the definition of the function w and the theorem conditions that w possesses the following properties:

$$w(x, t) \leqslant u(x, t) + \frac{\epsilon}{2}, \quad (x, t) \in \bar{\Omega}_T,$$
$$w(x_0, t_0) \geqslant u(x_0, t_0) = \epsilon + M \geqslant \epsilon + u(x, t) \geqslant \epsilon + w(x, t) - \frac{\epsilon}{2} \geqslant \frac{\epsilon}{2} + w(x, t), \quad (x, t) \in S_G^T.$$

The last property means that the function w cannot attain its maximum on the part S_G^T of the boundary of the domain Ω_T. If the maximum point of the function w over the domain $\bar{\Omega}_T$ is denoted by (x_1, t_1), then $x_1 \in G$, $0 < t_1 \leqslant T$ and

$$w(x_1, t_1) \geqslant w(x_0, t_0) \geqslant \epsilon + M > \epsilon. \tag{14}$$

Theorem 1 and the necessary conditions for the existence of the maximum in on open domain G lead then to the relations

$$\begin{cases} \big(D^{\alpha} w\big)(t_1) \geqslant 0, \\ \operatorname{grad} w|_{(x_1,t_1)} = 0, \quad \Delta w|_{(x_1,t_1)} \leqslant 0. \end{cases} \tag{15}$$

According to the definition of the function w, the function u satisfies the relation

$$u(x,t) = w(x,t) - \frac{\epsilon}{2}\frac{T-t}{T}, \quad (x,t) \in \bar{\Omega}_T. \tag{16}$$

The well-known formula ($0 < \alpha \leqslant 1$)

$$\big(D^{\alpha}\tau^{\beta}\big)(t) = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha}, \quad \beta > 0 \tag{17}$$

for the Caputo–Dzherbashyan fractional derivative leads to

$$\big(D^{\alpha} u\big)(t) = \big(D^{\alpha} w\big)(t) + \frac{\epsilon}{2T}\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{18}$$

Using the formulae (2), (4), (14)–(16), and (18), we arrive at the following chain of equalities and inequalities for the point (x_1, t_1):

$$\begin{aligned} &\big(D_t^{\alpha} u\big)(t_1) - \operatorname{div}(p \operatorname{grad} u) + qu - F \\ &\quad = \big(D^{\alpha} w\big)(t_1) + \frac{\epsilon}{2T}\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} - p\Delta v|_{(x_1,t_1)} - \big(\operatorname{grad} p|_{x_1}, \operatorname{grad} w|_{(x_1,t_1)}\big) + q\left(w - \frac{\epsilon}{2}\frac{T-t_1}{T}\right) - F \\ &\quad \geqslant \frac{\epsilon}{2T}\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + q\epsilon\left(1 - \frac{T-t_1}{2T}\right) > 0, \end{aligned}$$

that contradicts the condition of the theorem saying that the function u is a solution of Eq. (1). The obtained contradiction shows that the assumption made at the beginning of the theorem proof is wrong that proves the theorem.

Substituting $-u$ instead of u in the reasoning above, the minimum principle can be obtained.

Theorem 3. *Let a function $u \in C(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ be a solution of the generalized time-fractional diffusion equation* (1) *in the domain Ω_T and $F(x,t) \geqslant 0$, $(x,t) \in \Omega_T$. Then either $u(x,t) \geqslant 0$, $(x,t) \in \bar{\Omega}_T$ or the function u attains its negative minimum on the part S_G^T of the boundary of the domain Ω_T, i.e.,*

$$u(x,t) \geqslant \min_{(x,t)\in S_G^T} u(x,t), \quad \forall (x,t) \in \bar{\Omega}_T. \tag{19}$$

3. Uniqueness of the classical solution

In this section, the maximum principle proved in the previous section is applied to show that the problem (1), (5)–(6) possesses at most one classical solution and this solution—if it exists—continuously depends on the data given in the problem.

First, some a priory estimates for the solution norm are established.

Theorem 4. *Let u be a classical solution of the problem* (1), (5)–(6) *and F belong to the space $C(\bar{\Omega}_T)$ with the norm $M := \|F\|_{C(\bar{\Omega}_T)}$. Then the following estimate of the solution norm holds true*:

$$\|u\|_{C(\bar{\Omega}_T)} \leqslant \max\{M_0, M_1\} + \frac{T^{\alpha}}{\Gamma(1+\alpha)} M, \tag{20}$$

where

$$M_0 := \|u_0\|_{C(\bar{G})}, \qquad M_1 := \|v\|_{C(S\times[0,T])}. \tag{21}$$

To prove the theorem, we first introduce an auxiliary function w:

$$w(x,t) := u(x,t) - \frac{M}{\Gamma(1+\alpha)} t^{\alpha}, \quad (x,t) \in \bar{\Omega}_T.$$

Evidently, the function w is a classical solution of the problem (1), (5)–(6) with the functions $F_1(x,t) := F(x,t) - M - q(x)\frac{M}{\Gamma(1+\alpha)}t^{\alpha}$, $v_1(x,t) := v(x,t) - \frac{M}{\Gamma(1+\alpha)}t^{\alpha}$ instead of F and v, respectively. To get the expression for the function F_1, the formula (17) is used. The function F_1 satisfies the condition $F_1(x,t) \leqslant 0$, $(x,t) \in \bar{\Omega}_T$. Then the maximum principle applied to the classical solution w leads to the estimate

$$w(x,t) \leqslant \max\{M_0, M_1\}, \quad (x,t) \in \bar{\Omega}_T, \tag{22}$$

where the constants M_0, M_1 are defined as in (21). For the function u, we get

$$u(x,t) = w(x,t) + \frac{M}{\Gamma(1+\alpha)} t^\alpha \leqslant \max\{M_0, M_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)} M, \quad (x,t) \in \bar{\Omega}_T. \tag{23}$$

The minimum principle from Theorem 3 applied to the auxiliary function

$$w(x,t) := u(x,t) + \frac{M}{\Gamma(1+\alpha)} t^\alpha, \quad (x,t) \in \bar{\Omega}_T$$

leads to the estimate ($(x,t) \in \bar{\Omega}_T$)

$$u(x,t) \geqslant -\max\{M_0, M_1\} - \frac{T^\alpha}{\Gamma(1+\alpha)} M,$$

that together with the estimate (23) finishes the proof of the theorem.

We are now in the position to formulate and prove the following important theorem.

Theorem 5. *The problem* (1), (5)–(6) *possesses at most one classical solution. This solution continuously depends on the data given in the problem in the sense that if*

$$\|F - \tilde{F}\|_{C(\bar{\Omega}_T)} \leqslant \epsilon,$$
$$\|u_0 - \tilde{u}_0\|_{C(\bar{G})} \leqslant \epsilon_0, \qquad \|v - \tilde{v}\|_{C(S\times[0,T])} \leqslant \epsilon_1,$$

then the estimate

$$\|u - \tilde{u}\|_{C(\bar{\Omega}_T)} \leqslant \max\{\epsilon_0, \epsilon_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)} \epsilon \tag{24}$$

for the corresponding classical solutions u and $\tilde{u}$ holds true.

The uniqueness of the classical solution immediately follows from the fact that the homogeneous problem (1), (5)–(6), i.e., the problem with $F \equiv 0$, $u_0 \equiv 0$, and $v \equiv 0$ has only one classical solution, namely, $u(x,t) \equiv 0$, $(x,t) \in \bar{\Omega}_T$. The last statement is a simple consequence of the norm estimate (20) established in Theorem 4. The same estimate is used to prove the inequality (24). This time, it is applied to the function $u - \tilde{u}$ that is a classical solution of the problem (1), (5)–(6) with the functions $F - \tilde{F}$, $u_0 - \tilde{u}_0$, and $v - \tilde{v}$ instead of the functions F, u_0, and v, respectively.

4. Conclusions

In the paper, the time-fractional analog (with order of fractional differentiation between zero and one) of the maximum principle for spatially multi-dimensional linear parabolic differential equations with non-constant coefficients and a source term is stated and proved. To prove the maximum principle, an extremum principle for the Caputo–Dzherbashyan fractional derivative is first considered. Like in the case of the linear second-order parabolic partial differential equations, the maximum principle for the generalized time-fractional diffusion equations plays an outstanding role in the theory of these equations. In particular, in the paper it is applied to show that the initial-boundary-value problem for the generalized time-fractional diffusion equation possesses at most one classical solution and this solution continuously depends on the initial and boundary conditions, and the source term.

Of course, the maximum principle cannot be applied to show the existence of the solution; to do this, other methods are required. Because the operator L from the right-hand side of Eq. (1) is a positive definite and self-adjoint operator, its eigenvalues and eigenfunctions can be analyzed by following the standard methods. Thus the Fourier method of the variables separation seems to be the right one to construct a formal solution to the initial-boundary-value problems for the generalized time-fractional diffusion equation. In the general case however, this formal solution can be only interpreted as the generalized solution that can be sometimes reduced to a conventional function under certain additional conditions. All these questions and problems will be considered elsewhere.

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